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1976 J. Phys. A: Math. Gen. 9 1719

(http://iopscience.iop.org/0305-4470/9/10/023)

J. Phys. A: Math. Gen., Vol. 9, No. 10, 1976. Printed in Great Britain. © 1976

Propagation of the average carrier frequency of chirped pulses

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Received 24 November 1975, in final form 20 May 1976

Abstract. The modulation of frequency in resonant propagation can be characterized by the distance-dependent average carrier frequency of the pulse. We show that the propagation of this average frequency is governed by a conservation law specific for chirped pulses. We find a relation between the average frequency, the corresponding average wavevector and the energy of the pulse.

1. Introduction

The energy flow between different modes of the field may result in considerable changes of the spectrum of a light pulse during nonlinear propagation. Such effects are of primary importance in the process of formation of stable pulses. They also appear as small but cumulative corrections in long-distance steady-state propagation.

The frequency distribution of energy in the pulse depends on both the real envelope of the pulse and its phase when the phase varies with time. The variation of phase produces asymmetry in the energy distribution of the pulse around its carrier frequency. To describe the effects of the modulation of phase in resonant propagation Diels and Hahn (1973) introduced the notion of the *average carrier frequency* of the pulse. This average frequency can be simply expressed by the zeroth and first *spectral moments* of the pulse energy. The corresponding higher moments are small in the slowly varying envelope approximation (SVEA). Since a complete analysis of phase modulation described by a function of two variables is a complicated problem, the use of a set of a few moments depending on a single variable looks very promising. The present paper may be considered as a first step in formulating such a description.

We derive a differential conservation law specific for resonant propagation of chirped pulses and, as a consequence of it, we find the propagation law for the *first spectral moment* of the pulse energy. This law has a simple interpretation of conservation of the first moment of the total energy distribution in the system. The propagation law for the *average carrier frequency* follows as an immediate consequence from this law. Further, we find the propagation law for the corresponding average wavevector, define the *average 'phase velocity'* and establish a connection between the average quantities and the energy of the pulse generalized to include higher-order corrections in SVEA.

2. Equations of motion and basic definitions

Let $\mathscr{E}(z, t)$ and $\phi(z, t)$ denote the envelope and phase of the plane-wave electromagnetic pulse

$$\boldsymbol{E}(z, t)=\boldsymbol{e}\mathscr{E}(z, t)\,\mathrm{e}^{\mathrm{i}\phi(z,t)}\,\mathrm{e}^{\mathrm{i}(\omega_0 t-kz)}+\mathrm{CC} \tag{2.1}$$

where $\boldsymbol{e}$ is the circular polarization vector, ω_0 is the frequency of the carrier wave, $k=\omega_0/c$, c is the velocity of light in the host medium and CC stands for complex conjugate. Let $u(\gamma, z, t)$, $v(\gamma, z, t)$, $w(\gamma, z, t)$ denote the microscopic functions of two-level systems. The polarization due to an atom is determined by u and v in the usual way, while w is the population inversion. The parameter $\gamma=\Omega-\Omega_0$ measures the distance of an individual atomic frequency Ω from the central frequency of the atomic line Ω_0. The detuning is $\delta=\omega_0-\Omega_0$. For slowly varying $\mathscr{E}$ and ϕ the Bloch–Maxwell equations read (see e.g. Lamb 1971)

$$\frac{\partial u}{\partial t}=-(\gamma-\delta-\dot{\phi})v-\frac{u}{T_2}, \tag{2.2}$$

$$\frac{\partial v}{\partial t}=(\gamma-\delta-\dot{\phi})u+\frac{2\mathscr{P}}{\hbar}\mathscr{E}w-\frac{v}{T_2}, \tag{2.3}$$

$$\frac{\partial w}{\partial t}=-\frac{2\mathscr{P}}{\hbar}\mathscr{E}v, \tag{2.4}$$

$$\mathrm{D}\mathscr{E}+\tfrac{1}{2}c\sigma\mathscr{E}=\alpha\langle v\rangle, \tag{2.5}$$

$$\mathscr{E}\mathrm{D}\phi=-\alpha\langle u\rangle, \tag{2.6}$$

where

$$\mathrm{D}=\frac{\partial}{\partial t}+c\frac{\partial}{\partial z}, \qquad \dot{\phi}=\frac{\partial\phi}{\partial t},$$

$\alpha=2\pi N\omega_0\mathscr{P}$, N is the density of resonant atoms, $\mathscr{P}$ denotes the transition matrix element in the two-level system, σ accounts for broad-band linear losses or amplification, T_2 is the homogeneous relaxation time. Angular brackets denote averages over the distribution of frequencies in the atomic line, e.g.,

$$\langle u\rangle=\int_{-\infty}^{\infty} u(\gamma, z, t)g(\gamma)\,\mathrm{d}\gamma,$$

where $g(\gamma)$ is a symmetric distribution function.

The instantaneous carrier frequency of the pulse is $\omega_0+\dot{\phi}$. The *average carrier frequency*, as defined by Diels and Hahn (1973) is

$$\omega_{\mathrm{av}}=\omega_0+\dot{\phi}_{\mathrm{av}}, \tag{2.7}$$

where

$$\dot{\phi}_{\mathrm{av}}=\frac{1}{4\pi\mathscr{T}_0(z)}\int_{-\infty}^{\infty}\dot{\phi}(z, t)\mathscr{E}^2(z, t)\,\mathrm{d}t, \tag{2.8}$$

and

$$\mathscr{T}_0(z)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\mathscr{E}^2(z, t)\,\mathrm{d}t. \tag{2.9}$$

In terms of the Fourier transform $\boldsymbol{E}_\omega$ of the complex field (2.1) we can write

$$\mathscr{T}_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty} |\boldsymbol{E}_\omega|^2 \, \mathrm{d}\omega \tag{2.10}$$

whilst with a little manipulation we find

$$\mathscr{T}_0 \dot{\phi}_{\text{av}} = \frac{1}{4\pi}\int_{-\infty}^{\infty} (\omega - \omega_0)|\boldsymbol{E}_\omega|^2 \, \mathrm{d}\omega = \mathscr{T}_1. \tag{2.11}$$

$\mathscr{T}_1$ denotes the *first spectral moment* of the pulse. We note that $\mathscr{T}_1$ is of higher order in the SVEA than $\mathscr{T}_0$. In analogy with equations (2.10) and (2.11) we define the moments of the atomic energy distribution

$$W_0 = \tfrac{1}{2} N\hbar\omega_0 \int_{-\infty}^{\infty} w(\gamma, z, t) g(\gamma) \, \mathrm{d}\gamma, \tag{2.12}$$

$$W_1 = \tfrac{1}{2} N\hbar\omega_0 \int_{-\infty}^{\infty} w(\gamma, z, t) \gamma g(\gamma) \, \mathrm{d}\gamma. \tag{2.13}$$

According to equations (2.4) and (2.5) $\mathscr{T}_0$ satisfies the following propagation equation:

$$c\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{T}_0 = -\Delta W_0 - \tfrac{1}{2} c\sigma \mathscr{T}_0, \tag{2.14}$$

where

$$\Delta W_0 = W_0(z, \infty) - W_0(z, -\infty). \tag{2.15}$$

The *average wavevector* can be defined as

$$k_{\text{av}} = k - (\partial\phi/\partial z)_{\text{av}}, \tag{2.16}$$

where

$$\left(\frac{\partial\phi}{\partial z}\right)_{\text{av}} = \frac{1}{4\pi\mathscr{T}_0}\int_{-\infty}^{\infty} \frac{\partial\phi}{\partial z}\mathscr{E}^2 \, \mathrm{d}t. \tag{2.17}$$

3. Propagation laws

In order to find the propagation equation for $\dot{\phi}_{\text{av}}$, or the *first spectral moment* $\mathscr{T}_1$, we first consider the following differential expression:

$$\Gamma = \mathrm{D}(\mathscr{E}^2\dot{\phi}) + \frac{\partial}{\partial t}(\mathscr{E}^2 \mathrm{D}\phi). \tag{3.1}$$

Making use of equations (2.5) and (2.6) multiplied by $\mathscr{E}$ we can write

$$\Gamma = 2\alpha\dot{\phi}\mathscr{E}\langle v\rangle - 2\alpha\mathscr{E}\langle\partial u/\partial t\rangle - c\sigma\mathscr{E}^2\dot{\phi}. \tag{3.2}$$

From equation (2.2) we have

$$\langle\partial u/\partial t\rangle = \dot{\phi}\langle v\rangle - \langle(\gamma - \delta)v\rangle - \langle u\rangle/T_2, \tag{3.3}$$

and, from equation (2.4)

$$\mathscr{E}\langle(\gamma-\delta)v\rangle = -\frac{\hbar}{2\mathscr{P}}\frac{\partial}{\partial t}\langle(\gamma-\delta)w\rangle. \tag{3.4}$$

Finally equations (3.1–4) lead to

$$\mathrm{D}(\mathscr{E}^2\dot{\phi})+\frac{\partial}{\partial t}(\mathscr{E}^2\mathrm{D}\phi) = -2\pi N\hbar\omega_0\frac{\partial}{\partial t}\langle(\gamma-\delta)w\rangle - c\sigma\mathscr{E}^2\dot{\phi}-\frac{2\mathscr{E}^2\mathrm{D}\phi}{T_2}. \tag{3.5}$$

The above formula relates the variation of phase to the variation of $\mathscr{E}$ and w which determine the amount of energy in the system. In the absence of losses and relaxations equation (3.5) takes the form of a conservation law

$$\frac{\partial}{\partial t}\left(2\mathscr{E}^2\dot{\phi}+c\mathscr{E}^2\frac{\partial\phi}{\partial z}+2\pi N\hbar\omega_0\langle(\gamma-\delta)w\rangle\right)+c\frac{\partial}{\partial z}(\mathscr{E}^2\dot{\phi})=0. \tag{3.6}$$

This equation is independent of the energy conservation law (compare equation (3.10) below). It is trivially satisfied in the absence of chirping when $\dot{\phi}$ = constant (Michalska-Trautman 1975).

Integrating equation (3.5) over time we get, for a single pulse,

$$c\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{T}_1 = -\Delta W_1-(\Omega_0-\omega_0)\Delta W_0 - c\sigma\mathscr{T}_1-\frac{2\mathscr{T}_1}{T_2}-\frac{2c\mathscr{T}_0}{T_2}\left(\frac{\partial\phi}{\partial z}\right)_{\mathrm{av}} \tag{3.7}$$

where

$$\Delta W_1 = W_1(z,\infty)-W_1(z,-\infty).$$

In the absence of losses and relaxation equation (3.7) expresses the *conservation of the first moment* of the total energy distribution in the system calculated with respect to the carrier frequency ω_0 of the pulse.

Making use of equations (2.8), (2.9) and (2.12) we derive from equation (3.7) the propagation law for the *average carrier frequency*:

$$c\frac{\mathrm{d}\dot{\phi}_{\mathrm{av}}}{\mathrm{d}z} = -\frac{\Delta W_1}{\mathscr{T}_0}+\frac{\delta+\dot{\phi}_{\mathrm{av}}}{\mathscr{T}_0}\Delta W_0-\frac{1}{2}c\sigma\dot{\phi}_{\mathrm{av}}-\frac{2\dot{\phi}_{\mathrm{av}}}{T_2}-\frac{2c}{T_2}\left(\frac{\partial\phi}{\partial z}\right)_{\mathrm{av}} \tag{3.8}$$

or, more explicitly

$$\begin{aligned} c\frac{\mathrm{d}\dot{\phi}_{\mathrm{av}}}{\mathrm{d}z} = -\frac{1}{2}\frac{N\hbar\omega_0}{\mathscr{T}_0}\bigg(&\int_{-\infty}^{\infty}(w(\gamma,z,\infty)-w(\gamma,z,-\infty))\gamma g(\gamma)\,\mathrm{d}\gamma \\ &+[\Omega_0-(\omega_0+\dot{\phi}_{\mathrm{av}})]\int_{-\infty}^{\infty}(w(\gamma,z,\infty)-w(\gamma,z,-\infty))g(\gamma)\,\mathrm{d}\gamma\bigg)-\frac{1}{2}c\sigma\dot{\phi}_{\mathrm{av}} \\ &-\frac{2}{T_2}\left(\frac{\partial\phi}{\partial z}\right)_{\mathrm{av}}-\frac{2}{T_2}\dot{\phi}_{\mathrm{av}}. \end{aligned} \tag{3.9}$$

A similar equation, with the last term missing, was found by Diels and Hahn (1973).

The energy of the pulse does not depend on its phase in the lowest order of the slowly varying envelope approximation. This is seen from equation (2.14). However, when one takes into account corrections to the electromagnetic energy of higher order with respect to the expansion parameter $(\tau\omega_0)^{-1}$, where τ is the duration of the pulse, one finds the following energy conservation law:

$$\frac{1}{4\pi}\frac{\partial}{\partial t}\left[\mathscr{E}^2\left(1-\frac{\mathrm{D}\phi}{\omega_0}\right)\right]+c\frac{1}{4\pi}\frac{\partial}{\partial z}\left[\mathscr{E}^2\left(1-\frac{\mathrm{D}\phi}{\omega_0}\right)\right]$$
$$=-\frac{1}{2}N\hbar\frac{\partial}{\partial t}\langle\Omega w\rangle-\frac{c\sigma}{4\pi}\mathscr{E}^2-\frac{\mathscr{E}^2\mathrm{D}\phi}{2\pi T_2\omega_0}, \tag{3.10}$$

where $\Omega=\omega_0+\gamma-\delta$. In this equation $\mathscr{E}^2$ contains higher-order corrections to the envelope. The derivation of equation (3.10) and a detailed discussion of the orders of magnitude in the SVEA can be found in a forthcoming paper (Michalska-Trautman 1976). We denote by $\mathscr{T}(z)$ the integral over time of $\mathscr{E}^2$. The propagation law for the pulse energy can, therefore, be written in the form

$$c\omega_0\frac{\mathrm{d}T}{\mathrm{d}z}=-\Omega_0\Delta W_0-\Delta W_1-c\sigma\omega_0\mathscr{T}-\frac{2\mathscr{T}\dot{\phi}_{\mathrm{av}}}{T_2}-\frac{2c\mathscr{T}}{T_2}\left(\frac{\partial\phi}{\partial z}\right)_{\mathrm{av}} \tag{3.11}$$

where

$$T(z)=\mathscr{T}(z)\left[1-\frac{\dot{\phi}_{\mathrm{av}}}{\omega_0}-\frac{1}{k}\left(\frac{\partial\phi}{\partial z}\right)_{\mathrm{av}}\right]. \tag{3.12}$$

In this approximation the energy of the pulse depends on phase variation.

Equation (3.11) combined with equation (3.7) leads to the propagation law for the *average wavevector*:

$$c\frac{\mathrm{d}}{\mathrm{d}z}(\mathscr{T}(z)ck_{\mathrm{av}})=-2\Delta W_1-2\Omega_0\Delta W_0+\omega_0\Delta W_0. \tag{3.13}$$

The pulse energy T may also be written in the form

$$T(z)=\mathscr{T}(z)\left[1+\frac{\omega_{\mathrm{av}}}{\omega_0}\left(\frac{c}{V_{\mathrm{ph}}}-1\right)\right], \tag{3.14}$$

where

$$V_{\mathrm{ph}}=\omega_{\mathrm{av}}/k_{\mathrm{av}}$$

is the *average 'phase velocity'* of the pulse.

While this paper was being refereed, a paper by Deck and Lamb (1975) appeared, where a hierarchy of conservation laws for phase-dependent Maxwell–Bloch equations is derived. Our conservation law (3.6) corresponds to the second law of this hierarchy.

Acknowledgments

The author is grateful to Professor E L Hahn for an enlightening discussion and to an anonymous referee for a criticism of an earlier version of this paper which was submitted to the *Journal of Physics* in 1974.

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